8.1. Green & Stokes theorems I

8.1. Stokes theorem in 3d and Gauss theorem in 2d

The Stokes theorem is the FTC for a surface S (when the surface is just a region in a plane we call it <u>Green's theorem</u>).

A. We start with the formulation of the Stokes theorem for surfaces S in $\mathbb{R}^3$

B. We recall the general form of FTC theorem from 8.0. Then we think of Stokes theorem as its special case where the shape is a surface.

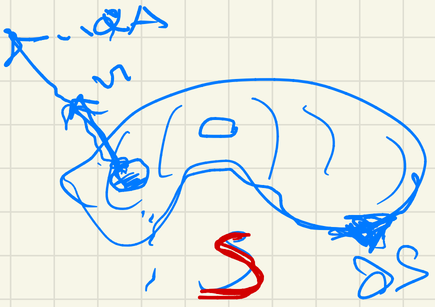
C. **Green's theorem:** we formulate it as a special case of Stokes when the surface S is a region in a plane.

Formulation of the

A. Stokes theorem:

FTC for surfaces

(vector valued area)



S an oriented surface.

$$\int_S \operatorname{curl}(F) \cdot d\vec{S} = \int_{\partial S} F \cdot dr$$

$d\vec{S} = n\,dA$

derivative of F

r = position vector

B. Recollection of FTC for a shape S and of the features one needs to make precise in each case

✓ (1) shape

~~(2) orientations~~ ✓

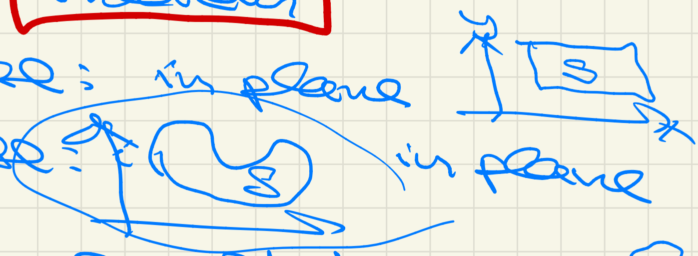
$\boxed{\text{(2) meaning of derivative}}$

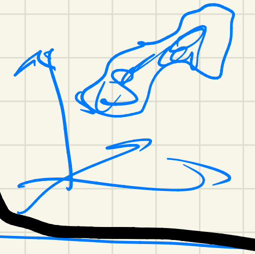
(3) meaning of integral: with respect to $d\ell$, dA, $d\vec{S}$ ✓

(3) der. of ψ = $\int_{\partial S} \psi$

C. Our Strategy for the proof of Stokes theorem:

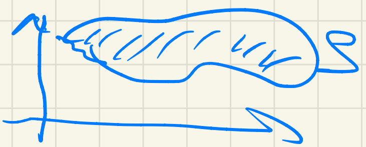
1. Very special case: in plane
2. Special case: in plane
3. General case: Stokes





D. We specialize Stokes theorem from space to plane:

Meaning of $\iint_S \mathrm{curl}(F)\cdot d\vec{S} = \oint_{\partial S} F\cdot dr$ in plane:

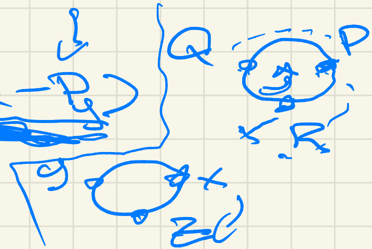


1. S is a region in $\mathbb{R}^2$
2. Our vector field F will be a planar one: $F = \langle P, Q\rangle$

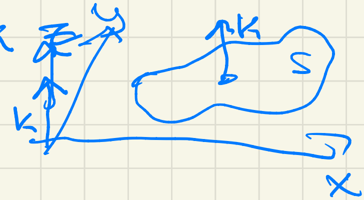
$F(x,y) = \langle P(x,y), Q(x,y)\rangle$

put F in space: $\hat{F}(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$

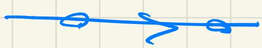
$$\mathrm{curl}(F) = \langle \underbrace{R_y}_{0} - \underbrace{Q_z}_{0},\ \underbrace{P_z}_{0} - \underbrace{R_x}_{0},\ Q_x - P_y\rangle$$



$$\operatorname{curl}(\vec{F}) = \langle 0, 0, Q_x - P_y \rangle$$
$$= (Q_x - P_y)\, k$$



I choose "standard orientation for regions": $n = \text{up}$



$n = k$

$$\iint_S \underbrace{\operatorname{curl}(F)}_{(Q_x - P_y)k} \cdot \underbrace{d\vec{S}}_{n\, dA} = \oint_{\partial S} \underbrace{F}_{\langle P, Q\rangle} \cdot \underbrace{d\vec{r}}_{d\begin{bmatrix} x \\ y \end{bmatrix}}$$

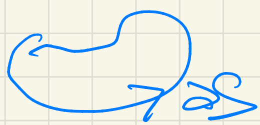
$(Q_x - P_y)k \cdot k\, dA$

Finally:

Stokes theorem in plane (=

$$\iint_S Q_x - P_y \, dA = \oint_{\partial S} P\,dx + Q\,dy$$

It is called the **Green's theorem**



$$(P x' + Q y')\, dt$$
$$= (P, Q) \begin{bmatrix} x' \\ y' \end{bmatrix} dt$$

Special case when surface lies in a plane:

{Green T.} $\iint_{\Omega} Q_x - P_y \, dx\,dy$

$\oint_{\partial\Omega} P\,dx + Q\,dy$

Announcement:

1st Step: check the case of a rectangle: